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Improved lumped-differential formulations and hybrid solution methods for drying in porous media

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Abstract

The present paper illustrates the construction of hybrid tools for the problem formulation and solution methodology towards the simulation of heat and mass transfer during drying in capillary porous media. First, a problem reformulation strategy is discussed, known as the coupled integral equations approach (CIEA), which offers improved lumped-differential formulations in different classes of problems, in comparison against classical lumping schemes, allowing for a reduction on the number of independent variables to be considered in specific formulations. Second, the generalized integral transform technique (GITT) is employed, as a hybrid numerical–analytical solution methodology for convection–diffusion problems. An example is provided related to drying in capillary porous cylindrical media as formulated by the two-dimensional Luikov's system of equations.

Keywords: Hybrid methods; Lumped systems; Integral transforms; Luikov equations; Porous media; Drying

1. Introduction

The present paper illustrates hybrid tools for the simulation process of drying in capillary porous media. Following the physical model construction, hybrid tools for development of the mathematical model and of the solution methodology are investigated. The hybrid nature is present in providing lumped-differential formulations, numerical–analytical methods, and symbolic–numerical computations.

First, a problem reformulation strategy is reviewed, based on Hermite integration schemes [1,2] and known as the coupled integral equations approach (CIEA) [3–10], which offers improved lumped-differential formulations in different classes of problems, in comparison with classical lumping schemes, allowing for a reduction on the number of independent variables to be considered in specific formulations, thus reducing simulation costs.

The phenomena of heat and mass transfer in capillary porous media has practical applications in several different areas including, among others, drying and the study of moisture migration in soils and construction materials [11,12]. For the mathematical modeling of such phenomena, Luikov [11,12] has proposed his widely known formulation, based on a system of coupled partial differential equations, which takes into account the effects of the temperature gradient on the moisture migration. A few approaches of analytical nature have been used for the solution of Luikov's equations in one-dimensional and multi-dimensional problems [13–18]. Nevertheless, several multidimensional heat transfer problems might involve small gradients along a specific spatial direction or even inside the whole body. A common engineering approach in such cases is to integrate the governing equations in the directions with smaller gradients. An approximate *lumped formulation* results when the gradients in such specific directions are neglected. However, an improved approximate formulation, which takes into account the gradients effects, can be obtained by using the present *coupled integral equations approach* (CIEA). The

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use of the CIEA can result in formulations as simple as those obtained with the classical lumped approach, but much more accurate, since the gradients are in fact approximated.

Afterwards, the well-established generalized integral transform technique (GITT) is employed, as a hybrid numerical–analytical solution methodology for diffusion and convection–diffusion problems [19–22]. The relative merits of such approach over purely numerical procedures, in light of its hybrid nature, are the automatic global accuracy control feature and the mild increase on computational costs for multidimensional nonlinear situations. The use of the *generalized integral transform technique* in drying problems with simple eigenvalue problems involving analytical eigenfunctions, can avoid the calculation of complex eigenvalues for this class of heat and mass transfer problem based on Luikov's formulation.

Thus, for the sake of illustration, in this paper we examine the solution of a two-dimensional drying problem in cylindrical coordinates. The coupled heat and mass transfer in the capillary-porous body is formulated with Luikov's equations. Temperature and moisture content gradients along the radial direction are supposed small, so that the governing equations are integrated in this direction. Both the lumped and the CIEA approximations are considered in this work to approximate the dependent variables at the surface of the cylinder. The resulting one-dimensional problem is solved with the generalized integral transform technique (GITT). The effects of lateral heat losses are then critically addressed.

2. Improved lumped-differential formulations: the coupled integral equations approach (CIEA)

The solution of multidimensional convection–diffusion problems, in the realm of practical applications, presents difficulties associated with a marked analytical involvement and/or considerable computational effort. Considering these facts, it becomes of interest for the engineering practice, to propose simpler formulations of the original partial differential systems, through a reduction of the number of independent variables in the multidimensional situations, by integration (averaging) of the full partial differential equations in one or more space variables, but retaining some information in the direction integrated out, provided by the related boundary conditions. Different levels of approximation in such mixed lumped-differential formulations can be used, starting from the plain and classical lumped system analysis, towards improved formulations, obtained through Hermite-type approximations for integrals [1,2]. Such approach has been already exploited in different heat and fluid flow problems [3–10], including phase change problems, extended surfaces (fins), anisotropic heat conduction, heat exchangers analysis, Navier–Stokes equations and drying problems themselves, and shall be reviewed within this section.

Hermite [1] developed a way of approximating an integral, based on the values of the integrand and its derivatives at the integration limits, in the form:

$$\int_{x_{i-1}}^{x_i} y(x)\,dx \cong \sum_{\nu=0}^{\alpha} C_\nu y_{i-1}^{(\nu)} + \sum_{\nu=0}^{\beta} D_\nu y_i^{(\nu)} \tag{1a}$$

where $y(x)$ and its derivatives $y^{(n)}(x)$ are defined for all $x \in (x_{i-1}, x_i)$. Furthermore, it is assumed that the numerical values of $y^{(\nu)}(x_{i-1}) \equiv y_{i-1}^{(\nu)}$ for $\nu = 0, 1, 2, \ldots, \alpha$ and $y^{(\nu)}(x_i) \equiv y_i^{(\nu)}$ for $\nu = 0, 1, 2, \ldots, \beta$, are available at the end points of the interval.

In such a manner, the integral of $y(x)$ is expressed as a linear combination of $y(x_{i-1})$, $y(x_i)$ and their derivatives, $y^{(\nu)}(x_{i-1})$ up to order $\nu = \alpha$, and $y^{(\nu)}(x_i)$ up to order $\nu = \beta$. This is called the $H_{\alpha,\beta}$ approximation. A detailed derivation for arbitrary α and β is presented in [2]. The resulting expression for the $H_{\alpha,\beta}$—approximation is given by:

$$\int_{x_{i-1}}^{x_i} y(x)\,dx = \sum_{\nu=0}^{\alpha} C_\nu(\alpha,\beta) h_i^{\nu+1} y_{i-1}^{(\nu)} + \sum_{\nu=0}^{\beta} C_\nu(\beta,\alpha)(-1)^\nu h_i^{\nu+1} y_i^{(\nu)} + O\big(h_i^{\alpha+\beta+3}\big) \tag{1b}$$

where

$$h_i = x_i - x_{i-1} \tag{1c}$$

$$C_\nu(\alpha,\beta) = \frac{(\alpha+1)!(\alpha+\beta+1-\nu)!}{(\nu+1)!(\alpha-\nu)!(\alpha+\beta+2)!} \tag{1d}$$

In the following example, we consider just the two approximations, $H_{0,0}$ and $H_{1,1}$, given by:

$$H_{0,0} \to \int_0^h y(x)\,dx \cong \frac{h}{2}\big(y(0) + y(h)\big) \tag{2}$$

$$H_{1,1} \to \int_0^h y(x)\,dx \cong \frac{h}{2}\big(y(0) + y(h)\big) + \frac{h^2}{12}\big(y'(0) - y'(h)\big) \tag{3}$$

which correspond, respectively, to the well-known trapezoidal and corrected trapezoidal integration rules. The respective expressions for the errors in the approximations $H_{0,0}$ and $H_{1,1}$ are written as:

$$E_{0,0} = -\frac{h^3}{12} y''(\eta), \quad \eta \in (0, h) \tag{4a}$$

$$E_{1,1} = +\frac{h^5}{720} y^{iv}(\xi), \quad \xi \in (0, h) \tag{4b}$$

and can be employed to compose the final error expression in the desired averaged potential, which may then be bounded for *a priori* error analysis.

Under certain boundary conditions and other characteristics of the diffusion process, the fully differential formulations can be markedly simplified through a reduction of the number of independent variables involved. Thus, one or more space variables may be integrated out in the original differential formulations, yielding, through the use of expressions such as Eqs. (2), (3), approximate formulations that retain local information on the remaining coordinates, and averaged information in the directions eliminated through integration.

Despite all the progress achieved in the computational solution of drying problems formulated by the original set of Luikov's equations, these methodologies are still quite too complex for engineering-type work in the realm of applications. In particular due to the multidisciplinary aspects of this physical problem, appearing within various sciences branches where a profound mathematical background might not be *a priori* required, the development of simplified formulations becomes of major relevance. One such possibility of simplification is the classical lumped system analysis, based on the assumption of uniform distribution of the associated potentials over the whole problem domain or along selected coordinates. The alternative technique of producing approximate formulations described in this section, based on the use of Hermite-type approximations for integrals, is here examined. Therefore, the present example is aimed at illustrating improved lumped-differential formulations developed in the context of drying problems [3,8,10], starting from the Luikov system of partial differential equations [11,12]. The integral transform solution of the original Luikov system provides the reference results [13–17] to illustrate the accuracy and applicability limits of the approximate formulations.

The physical problem under picture involves a cylindrical capillary porous medium of radius R_0 and length l, initially at uniform temperature and uniform moisture content. One of the boundaries, which is impervious to moisture transfer, is put in contact with a heater. The other boundary is put in contact with the dry surrounding air, thus resulting in a convective boundary condition for both the temperature and the moisture content. The lateral surface of the cylinder is also supposed to be impervious to mass transfer, but heat losses at this boundary are taken into account through a convective boundary condition. The system of equations proposed by Luikov, for the modeling of such physical problem involving the drying of a capillary porous media, can be written in dimensionless form as [11,12]:

$$\frac{\partial\theta(R,Z,\tau)}{\partial\tau} = \alpha\frac{\partial^2\theta(R,Z,\tau)}{\partial Z^2} - \beta\frac{\partial^2\phi(R,Z,\tau)}{\partial Z^2} + \frac{r_a^2\alpha}{R}\frac{\partial}{\partial R}\left[R\frac{\partial\theta(R,Z,\tau)}{\partial R}\right] - \frac{r_a^2\beta}{R}\frac{\partial}{\partial R}\left[R\frac{\partial\phi(R,Z,\tau)}{\partial R}\right]$$
$$\text{in } 0<R<1 \text{ and } 0<Z<1,\ \text{for } \tau>0 \tag{5a}$$

$$\frac{\partial\phi(R,Z,\tau)}{\partial\tau} = Lu\frac{\partial^2\phi(R,Z,\tau)}{\partial Z^2} - Lu\,Pn\frac{\partial^2\theta(R,Z,\tau)}{\partial Z^2} + \frac{r_a^2 Lu}{R}\frac{\partial}{\partial R}\left[R\frac{\partial\phi(R,Z,\tau)}{\partial R}\right] - \frac{r_a^2 Lu\,Pn}{R}\frac{\partial}{\partial R}\left[R\frac{\partial\theta(R,Z,\tau)}{\partial R}\right]$$
$$\text{in } 0<R<1 \text{ and } 0<Z<1,\ \text{for } \tau>0 \tag{5b}$$

$$\theta(R,Z,0)=0 \quad \text{for } \tau=0,\ \text{in } 0<R<1 \text{ and } 0<Z<1 \tag{5c}$$

$$\phi(R,Z,0)=0 \quad \text{for } \tau=0,\ \text{in } 0<R<1 \text{ and } 0<Z<1 \tag{5d}$$

$$\frac{\partial\theta(0,Z,\tau)}{\partial R}=0, \quad \text{at } R=0 \text{ and } 0 \text{ for } \tau>0 \tag{5e}$$

$$\frac{\partial\phi(0,Z,\tau)}{\partial R}=0 \quad \text{at } R=0 \text{ and } 0 \text{ for } \tau>0 \tag{5f}$$

$$\frac{\partial\theta(1,Z,\tau)}{\partial R} - Bi_{\text{qr}}\big[1-\theta(1,Z,\tau)\big]=0 \quad \text{at } R=1 \text{ and } 0<Z<1 \text{ for } \tau>0 \tag{5g}$$

$$\frac{\partial\phi(1,Z,\tau)}{\partial R} = Pn\frac{\partial\theta(1,Z,\tau)}{\partial R} \quad \text{at } R=1 \text{ and } 0<Z<1 \text{ for } \tau>0 \tag{5h}$$

$$\frac{\partial\theta(R,0,\tau)}{\partial Z} = -Q \quad \text{at } Z=0 \text{ and } 0<R<1,\ \text{for } \tau>0 \tag{5i}$$

$$\frac{\partial\phi(R,0,\tau)}{\partial Z} = -Pn\,Q \quad \text{at } Z=0 \text{ and } 0<R<1,\ \text{for } \tau>0 \tag{5j}$$

$$\frac{\partial\theta(R,1,\tau)}{\partial Z} - Bi_q\big[1-\theta(R,1,\tau)\big] + (1-\varepsilon)Ko\,Lu\,Bi_m\big[1-\phi(R,1,\tau)\big]=0, \quad \text{at } Z=1 \text{ and } 0<R<1,\ \text{for } \tau>0 \tag{5k}$$

$$\frac{\partial\phi(R,1,\tau)}{\partial Z} + Bi_m^*\phi(R,1,\tau) = Bi_m^* - Pn\,Bi_q\big[\theta(R,1,\tau)-1\big], \quad \text{at } Z=1 \text{ and } 0<R<1,\ \text{for } \tau>0 \tag{5l}$$

The various dimensionless groups appearing above are defined as

$$\theta(R,Z,\tau) = \frac{T(r,z,t)-T_0}{T_s-T_0} \tag{6a}$$

$$\phi(R,Z,\tau) = \frac{u_0-u(r,z,t)}{u_0-u_s} \tag{6b}$$

$$Q = \frac{ql}{k(T_s-T_0)} \tag{6c}$$

$$\tau = \frac{at}{l^2} \tag{6d}$$

$$Lu = \frac{a_m}{a} \tag{6e}$$

$$Pn = \delta\frac{T_s-T_0}{u_0-u_s} \tag{6f}$$

$$Bi_q = \frac{hl}{k} \tag{6g}$$

$$Bi_m = \frac{h_m l}{k_m} \tag{6h}$$

$$Ko = \frac{\lambda}{c}\frac{u_0-u_s}{T_s-T_0} \tag{6i}$$

$$Bi_{\text{qr}} = \frac{h_r r_0}{k} \tag{6j}$$

$$r_a = \frac{l}{r_0} \tag{6k}$$

$$R = \frac{r}{r_0} \tag{6l}$$

$$Z = \frac{z}{l} \tag{6m}$$

$$Bi_m^* = Bi_m\big[1-(1-\varepsilon)Pn\,Ko\,Lu\big] \tag{6n}$$

$$\alpha = 1 + \varepsilon\, Ko\, Lu\, Pn \tag{6o}$$

$$\beta = \varepsilon\, Ko\, Lu \tag{6p}$$

where a is the thermal diffusivity of the porous medium, a_m is the moisture diffusivity in the porous medium, c is the specific heat of porous medium, h and h_r are the heat transfer coefficients at the top and lateral surfaces, respectively, h_m is the mass transfer coefficient, k is the thermal conductivity, k_m is the moisture conductivity, l is the thickness of porous medium, q is the prescribed heat flux, λ is the latent heat of evaporation of water, T_s is the temperature of the surrounding air, T_0 is the uniform initial temperature in the medium, u_s is the moisture content of the surrounding air, u_0 is the uniform initial moisture content in the medium, δ is the thermogradient coefficient and ε is the phase conversion factor. *Lu*, *Pn* and *Ko* denote the Luikov, Posnov and Kossovitch numbers, respectively.

In order to derive the approximate formulations addressed in this work, we integrate equations (5) along the radial direction. By substituting into the resulting expressions the boundary conditions (5e–h) and by using the following definitions of average temperature and moisture content at each cross section:

$$\widehat{\theta}(Z,\tau) = 2\int_0^1 R\theta(R,Z,\tau)\,dR \tag{7a}$$

$$\widehat{\phi}(Z,\tau) = 2\int_0^1 R\phi(R,Z,\tau)\,dR \tag{7b}$$

we can write the formulation for the coupled heat and mass transfer problem under picture here as:

$$\frac{\partial\widehat{\theta}(Z,\tau)}{\partial\tau} = \alpha\frac{\partial^2\widehat{\theta}(Z,\tau)}{\partial Z^2} - \beta\frac{\partial^2\widehat{\phi}(Z,\tau)}{\partial Z^2} - 2r_a^2 Bi_{\mathrm{qr}}\theta(1,Z,\tau) + 2r_a^2 Bi_{\mathrm{qr}} \quad \text{in } 0<Z<1,\ \text{for } \tau>0 \tag{8a}$$

$$\frac{\partial\widehat{\phi}(Z,\tau)}{\partial\tau} = Lu\frac{\partial^2\widehat{\phi}(Z,\tau)}{\partial Z^2} - Lu\,Pn\frac{\partial^2\widehat{\theta}(Z,\tau)}{\partial Z^2} \quad \text{in } 0<Z<1,\ \text{for } \tau>0 \tag{8b}$$

$$\widehat{\theta}(Z,0) = 0, \quad \text{for } \tau=0,\ \text{in } 0<Z<1 \tag{8c}$$

$$\widehat{\phi}(Z,0) = 0, \quad \text{for } \tau=0,\ \text{in } 0<Z<1 \tag{8d}$$

$$\frac{\partial\widehat{\theta}(0,\tau)}{\partial Z} = -Q, \quad \text{at } Z=0 \text{ for } \tau>0 \tag{8e}$$

$$\frac{\partial\widehat{\phi}(0,\tau)}{\partial Z} = -Pn\,Q, \quad \text{at } Z=0 \text{ for } \tau>0 \tag{8f}$$

$$\frac{\partial\widehat{\theta}(1,\tau)}{\partial Z} - Bi_q\big[1-\widehat{\theta}(1,\tau)\big] + (1-\varepsilon)Ko\,Lu\,Bi_m\big[1-\widehat{\phi}(1,\tau)\big] = 0 \quad \text{at } Z=1 \text{ for } \tau>0 \tag{8g}$$

$$\frac{\partial\widehat{\phi}(1,\tau)}{\partial Z} + Bi_m^*\widehat{\phi}(1,\tau) = Bi_m^* - Pn\,Bi_q\big[\widehat{\theta}(1,\tau)-1\big] \quad \text{at } Z=1 \text{ for } \tau>0 \tag{8h}$$

We note in Eq. (8a) that such a formulation for the problem involves the temperature at the lateral surface of the body, besides the average temperature and average moisture content at each cross section. Two different approaches are used here to approximate $\theta(1,Z,\tau)$, as described next.

2.1. Classical lumped approach

In the traditional lumped approach, gradients inside the body along the radial direction are completely neglected. Therefore, we can approximate the temperature at the lateral surface by the average temperature, that is,

$$\theta(1,Z,\tau) = \hat{\theta}(Z,\tau) \tag{9}$$

With the use of the approximation given by Eq. (9), problem (8) only involves as dependent variables the average temperature and average moisture content at each cross section.

2.2. Coupled integral equations approach

An improved approximate formulation can be obtained, by using Hermite integrals to write the temperature at the lateral surface of the body in terms of the average temperature at each cross section.

In this work, we use the $H_{1,1}$ expression in order to approximate the average temperature defined by Eq. (7a) and the $H_{0,0}$ expression to approximate the integral of the temperature gradient along the radial direction, that is,

$$\widehat{\theta}(Z,\tau) = 2\int_0^1 R\theta(R,Z,\tau)\,dR \cong \theta(1,Z,\tau) + \frac{1}{6}\left[\frac{\partial}{\partial R}\big[R\theta(R,Z,\tau)\big]\Big|_{R=0} - \frac{\partial}{\partial R}\big[R\theta(R,Z,\tau)\big]\Big|_{R=1}\right] \tag{10a}$$

$$\int_0^1 \frac{\partial\theta(R,Z,\tau)}{\partial R}\,dR = \big[\theta(1,Z,\tau)-\theta(0,Z,\tau)\big] \cong \frac{1}{2}\left[\frac{\partial\theta(1,Z,\tau)}{\partial R} + \frac{\partial\theta(0,Z,\tau)}{\partial R}\right] \tag{10b}$$

Eqs. (10a), (10b) are then solved in order to obtain the following expression for the temperature at the lateral surface of the body

$$\theta(1,Z,\tau) = \frac{4}{4+Bi_{\mathrm{qr}}}\left(\widehat{\theta}(Z,\tau) + \frac{Bi_{\mathrm{qr}}}{4}\right) \tag{11}$$

By substituting Eq. (11) into Eq. (8a) and rearranging, we obtain a formulation for the present problem similar to that obtained with the lumped approach, except for the Biot number at the lateral surface. For the CIEA, a modified Biot number appears in the formulation. It is defined as:

$$Bi_{\mathrm{qr}}^* = \frac{4Bi_{\mathrm{qr}}}{4+Bi_{\mathrm{qr}}} \tag{12}$$

Therefore, we can write the approximate formulation for problem (5) as

$$\frac{\partial\widehat{\theta}(Z,\tau)}{\partial\tau}=\alpha\frac{\partial^2\widehat{\theta}(Z,\tau)}{\partial Z^2}-\beta\frac{\partial^2\widehat{\phi}(Z,\tau)}{\partial Z^2}-\eta\widehat{\theta}(Z,\tau)+\eta \quad \text{in } 0<Z<1,\ \text{for } \tau>0 \tag{13a}$$

$$\frac{\partial\widehat{\phi}(Z,\tau)}{\partial\tau}=Lu\frac{\partial^2\widehat{\phi}(Z,\tau)}{\partial Z^2}-Lu\,Pn\frac{\partial^2\widehat{\theta}(Z,\tau)}{\partial Z^2} \quad \text{in } 0<Z<1,\ \text{for } \tau>0 \tag{13b}$$

$$\widehat{\theta}(Z,0)=0, \quad \text{for } \tau=0,\ \text{in } 0<Z<1 \tag{13c}$$

$$\widehat{\phi}(Z,0)=0, \quad \text{for } \tau=0,\ \text{in } 0<Z<1 \tag{13d}$$

$$\frac{\partial\widehat{\theta}(0,\tau)}{\partial Z}=-Q, \quad \text{at } Z=0,\ \text{for } \tau>0 \tag{13e}$$

$$\frac{\partial\widehat{\phi}(0,\tau)}{\partial Z}=-Pn\,Q, \quad \text{at } Z=0,\ \text{for } \tau>0 \tag{13f}$$

$$\frac{\partial\widehat{\theta}(1,\tau)}{\partial Z}-Bi_q\big[1-\widehat{\theta}(1,\tau)\big]+(1-\varepsilon)Ko\,Lu\,Bi_m\big[1-\widehat{\phi}(1,\tau)\big]=0 \quad \text{at } Z=1,\ \text{for } \tau>0 \tag{13g}$$

$$\frac{\partial\widehat{\phi}(1,\tau)}{\partial Z}+Bi_m^*\widehat{\phi}(1,\tau)=Bi_m^*-Pn\,Bi_q\big[\widehat{\theta}(1,\tau)-1\big] \quad \text{at } Z=1,\ \text{for } \tau>0 \tag{13h}$$

where, for the lumped approach we have

$$\eta=2r_a^2Bi_{\mathrm{qr}} \tag{14a}$$

and for the CIEA based on the $H_{1,1}$, $H_{0,0}$ approximations given by Eqs. (10a), (10b) we have:

$$\eta=2r_a^2Bi_{\mathrm{qr}}^* \tag{14b}$$

Therefore, the use of the approximation for $\theta(1,Z,\tau)$, obtained with the CIEA does not introduce any additional complexity into the formulation for the problem, as compared to the lumped approach. However, more accurate results are expected with the CIEA instead of the lumped approach, since the radial gradients in the body are now taken care of, through the approximation given by Eq. (11), instead of being neglected.

We note that the use of the $H_{0,0}$ approximation for the average temperature results in a formulation identical to that obtained with the lumped analysis and, hence, is not discussed here.

3. Hybrid methods: the generalized integral transform technique (GITT)

Within the last two decades, the classical integral transform method [18] gained a hybrid numerical–analytical structure, offering user controlled accuracy and quite efficient computational performance for a wide variety of *a priori* nontransformable problems [19–22], including the nonlinear formulations of interest in heat and fluid flow applications. Besides being an alternative computational method on itself, this hybrid approach is particularly well suited for benchmarking purposes, in light of its automatic error control feature, retaining the same characteristics of a purely analytical solution. In addition to the straightforward error control and estimation, an outstanding aspect of this method is the direct extension to multidimensional situations, with a moderate increase in computational effort with respect to one-dimensional applications. Again, the hybrid nature is responsible for this behavior, since the analytical part in the solution procedure is employed over all but one independent variable, and the numerical task is always reduced to the integration of an ordinary differential system in this one single independent variable.

The example selected for illustration thus involves simultaneous heat and mass transfer during drying of a capillary porous body under the Luikov model, according to the one-dimensional problem formulation that results from the CIEA reformulation previously developed.

We use in this work the GITT for the solution of the one-dimensional problem (13), following the approach advanced in [13,14]. In order to reduce the effects of the non-homogeneities on the convergence of the series solution, we rewrite the solution of problem (13) as

$$\widehat{\theta}(Z,\tau)=\widehat{\theta}_s(Z)+\widehat{\theta}_h(Z,\tau) \tag{15a}$$

$$\widehat{\phi}(Z,\tau)=\widehat{\phi}_s(Z)+\widehat{\phi}_h(Z,\tau) \tag{15b}$$

where the steady-state problem is separated from the analysis:

$$\alpha\frac{d^2\widehat{\theta}_s(Z)}{dZ^2}=\beta\frac{d^2\widehat{\phi}_s(Z)}{dZ^2}+\eta\widehat{\theta}_s(Z)-\eta \quad \text{in } 0<Z<1 \tag{16a}$$

$$\frac{d^2\widehat{\phi}_s(Z)}{dZ^2}=Pn\frac{d^2\widehat{\theta}_s(Z)}{dZ^2} \quad \text{in } 0<Z<1 \tag{16b}$$

$$\frac{d\theta_s(0)}{\partial Z}=-Q, \quad \text{at } Z=0 \tag{16c}$$

$$\frac{d\phi_s(0)}{dZ}=-Pn\,Q, \quad \text{at } Z=0 \tag{16d}$$

$$\frac{d\theta_s(1)}{dZ}+Bi_q\theta_s(1)=Bi_q-(1-\varepsilon)Ko\,Lu\,Bi_m\big[1-\phi_s(1)\big] \quad \text{at } Z=1 \tag{16e}$$

$$\frac{d\phi_s(1)}{dZ}+Bi_m^*\phi_s(1)=Bi_m^*-Pn\,Bi_q\big[\theta_s(1)-1\big] \quad \text{at } Z=1 \tag{16f}$$

System (16) is readily solved symbolically through the use of the Mathematica package [23]. By substituting equations (15a), (15b) into Eqs. (13) and using equations (16), we obtain the homogeneous problem as

$$\frac{\partial\widehat{\theta}_h(Z,\tau)}{\partial\tau}=\alpha\frac{\partial^2\widehat{\theta}_h(Z,\tau)}{\partial Z^2}-\beta\frac{\partial^2\widehat{\phi}_h(Z,\tau)}{\partial Z^2}-\eta\widehat{\theta}_h(Z,\tau) \quad \text{in } 0<Z<1,\ \text{for } \tau>0 \tag{17a}$$

$$\frac{\partial\widehat{\phi}_h(Z,\tau)}{\partial\tau}=Lu\frac{\partial^2\widehat{\phi}_h(Z,\tau)}{\partial Z^2}-Lu\,Pn\frac{\partial^2\widehat{\theta}_h(Z,\tau)}{\partial Z^2} \quad \text{in } 0<Z<1,\ \text{for } \tau>0 \tag{17b}$$

$$\widehat{\theta}_h(Z,0)=-\widehat{\theta}_s(Z), \quad \text{for } \tau=0,\ \text{in } 0<Z<1 \tag{17c}$$

$$\widehat{\phi}_h(Z,0)=-\widehat{\phi}_s(Z), \quad \text{for } \tau=0,\ \text{in } 0<Z<1 \tag{17d}$$

$$\frac{\partial\widehat{\theta}_h(0,\tau)}{\partial Z}=0, \quad \text{at } Z=0,\ \text{for } \tau>0 \tag{17e}$$

$$\frac{\partial\widehat{\phi}_h(0,\tau)}{\partial Z}=0, \quad \text{at } Z=0,\ \text{for } \tau>0 \tag{17f}$$

$$\frac{\partial\widehat{\theta}_h(1,\tau)}{\partial Z}+Bi_q\widehat{\theta}_h(1,\tau)=(1-\varepsilon)Ko\,Lu\,Bi_m\widehat{\phi}_h(1,\tau)$$
at $Z=1$, for $\tau>0$ (17g)

$$\frac{\partial\widehat{\phi}_h(1,\tau)}{\partial Z}+Bi_m^*\widehat{\phi}_h(1,\tau)=-Pn\,Bi_q\widehat{\theta}_h(1,\tau)$$
at $Z=1$, for $\tau>0$ (17h)

The integral transform/inverse formula pairs for temperature and moisture content are defined, respectively, as

$$\bar{\widehat{\theta}}_i(\tau)=\frac{1}{N_i^{1/2}}\int_0^1\varphi_i(Z)\widehat{\theta}_h(Z,\tau)\,dZ \tag{18a}$$

$$\widehat{\theta}_h(Z,\tau)=\sum_{i=1}^{\infty}\frac{1}{N_i^{1/2}}\varphi_i(Z)\bar{\widehat{\theta}}_i(\tau) \tag{18b}$$

and

$$\bar{\widehat{\phi}}_i(\tau)=\frac{1}{P_i^{1/2}}\int_0^1\Gamma_i(Z)\widehat{\phi}_h(Z,\tau)\,dZ \tag{19a}$$

$$\widehat{\phi}_h(Z,\tau)=\sum_{i=1}^{\infty}\frac{1}{P_i^{1/2}}\Gamma_i(Z)\bar{\widehat{\phi}}_i(\tau) \tag{19b}$$

The eigenfunctions and normalization integral for temperature are given, respectively, by:

$$\varphi_i(Z)=\cos(\gamma_i Z) \tag{20a}$$

$$N_i=\frac{1}{2}\left[1+\frac{Bi_q}{\gamma_i^2+Bi_q^2}\right] \tag{20b}$$

where the eigenvalues are the positive roots of

$$(\gamma_i)\tan(\gamma_i)=Bi_q \tag{20c}$$

Similarly, we have for the moisture content

$$\Gamma_i(Z)=\cos(\xi_i Z) \tag{21a}$$

$$P_i=\frac{1}{2}\left[1+\frac{Bi_m^*}{\xi_i^2+Bi_m^{*2}}\right] \tag{21b}$$

with eigenvalues given by the positive roots of

$$(\xi_i)\tan(\xi_i)=Bi_m^* \tag{21c}$$

The integral transformation of problem (17) results on the following system of coupled ordinary differential equations:

$$\frac{d\bar{\widehat{\theta}}_i(\tau)}{d\tau}+\left(\alpha\gamma_i^2+\eta\right)\bar{\widehat{\theta}}_i(\tau)-\beta\sum_{j=1}^{\infty}A_{ij}^*\bar{\widehat{\phi}}_j(\tau)$$
$$=\frac{\varphi_i(1)}{N_i^{1/2}}Ko\,Lu\big[(Bi_m-\varepsilon Bi_q)\widehat{\phi}_h(1,\tau)$$
$$+\varepsilon\,Pn\,Bi_q\widehat{\theta}_h(1,\tau)\big] \tag{22a}$$

$$\frac{d\bar{\widehat{\phi}}_i(\tau)}{d\tau}+Lu\,\xi_i^2\bar{\widehat{\phi}}_i(\tau)-Lu\,Pn$$
$$=-\frac{\Gamma_i(1)}{P_i^{1/2}}Lu\,Pn\big[Bi_m^*\widehat{\theta}_h(1,\tau)$$
$$+(1-\varepsilon)Ko\,Lu\,Bi_m\widehat{\phi}_h(1,\tau)\big] \tag{22b}$$

where

$$\widehat{\theta}_h(1,\tau)=\sum_{j=1}^{\infty}\frac{1}{N_j^{1/2}}\varphi_j(1)\bar{\widehat{\theta}}_j(\tau) \tag{22c}$$

$$\widehat{\phi}_h(1,\tau)=\sum_{j=1}^{\infty}\frac{1}{P_j^{1/2}}\Gamma_j(1)\bar{\widehat{\phi}}_j(\tau) \tag{22d}$$

$$\bar{f}_j=\frac{1}{N_j^{1/2}}\int_0^1\varphi_j(Z)\,dZ \tag{22e}$$

$$\bar{f}_j^*=\frac{1}{P_j^{1/2}}\int_0^1\Gamma_j(Z)\,dZ \tag{22f}$$

$$A_{ij}^*=\frac{1}{N_i^{1/2}P_j^{1/2}}\int_0^1\varphi_i(Z)\Gamma_j(Z)\,dZ \tag{22g}$$

$$B_{ij}^*=\frac{1}{N_j^{1/2}P_i^{1/2}}\int_0^1\varphi_j(Z)\Gamma_i(Z)\,dZ \tag{22h}$$

The solution for the system (22), truncated to a sufficiently large order to reach convergence, is obtained with the subroutine DIVPAG of the IMSL [24], while the system coefficients are symbolically determined with the *Mathematica* software system [23]. Then, the average temperature and moisture content in the region can be computed by using the inverse formula given by Eqs. (18b), (19b).

4. Results and discussion

We focus the analysis below on the effects of the Biot number in the radial direction on the approximate solutions obtained *via* both the lumped and $H_{1,1}/H_{0,0}$ approaches. Thus, computer runs were performed for different values of the radial Biot number, $Bi_{\mathrm{qr}}=0$, 0.1, 1.0 and 10. Other parameters of importance for the analysis were taken as: $Lu=0.4$, $Pn=0.6$, $Ko=5.0$, $Bi_q=Bi_m=2.5$, $\varepsilon=0.2$ and $Q=0.9$. We first illustrate the convergence behavior of the proposed eigenfunction expansions for the improved lumped solution, obtained through integral transforms. Therefore, Tables 1a–1f show the convergence behavior of both the temperature and moisture content profiles, for $Bi_{\mathrm{qr}}=0.1$, 1.0 and 10, and dimensionless times $\tau=0.2$ and 0.5, for three longitudinal positions, $Z=0.1$, 0.5, and 0.9. Increasing truncation orders in the expansions, N, here taken identical to both temperature and moisture fields, demonstrate the full convergence to four decimal digits in all cases considered. Also, truncation orders as low as $N=10$ are considered to illustrate that fairly reasonable results for most practical purposes might be obtained when computer effort reduction is at a premium. On the other hand, three digits convergence is already evident from the columns referring to 100 terms in the expansions.

Next, we present a series of results, Tables 2a–2f, again for $Bi_{\mathrm{qr}}=0.1$, 1.0 and 10, and dimensionless times $\tau=0.2$ and 0.5, for three longitudinal positions, $Z=0.1$, 0.5, and 0.9,

Table 1a
Convergence behavior of moisture content and temperature for $Bi_{qr} = 0.1$, $\tau = 0.2$

N	$\theta(Z,\tau)$					$\phi(Z,\tau)$				
	100	200	300	350	390	100	200	300	350	390
$Z = 0.1$	0.3017	0.3019	0.3020	0.3020	0.3020	0.0773	0.0772	0.0772	0.0772	0.0772
$Z = 0.5$	0.0597	0.0601	0.0602	0.0603	0.0603	0.0519	0.0518	0.0517	0.0517	0.0517
$Z = 0.9$	−0.0005	0.00002	0.0002	0.0003	0.0003	0.3333	0.3327	0.3325	0.3324	0.3324

Table 1b
Convergence behavior of moisture content and temperature for $Bi_{qr} = 0.1$ and $\tau = 0.5$

N	$\theta(Z,\tau)$					$\phi(Z,\tau)$				
	100	200	300	350	390	100	200	300	350	390
$Z = 0.1$	0.5344	0.5349	0.5351	0.5351	0.5352	0.2000	0.1998	0.1997	0.1997	0.1997
$Z = 0.5$	0.3098	0.3103	0.3104	0.3105	0.3105	0.1968	0.1964	0.1963	0.1963	0.1963
$Z = 0.9$	0.2604	0.2606	0.2607	0.2608	0.2608	0.4773	0.4765	0.4762	0.4761	0.4761

Table 1c
Convergence behavior of moisture content and temperature for $Bi_{qr} = 1$ and $\tau = 0.2$

N	$\theta(Z,\tau)$					$\phi(Z,\tau)$				
	10	100	200	300	330	10	100	200	300	330
$Z = 0.1$	0.5005	0.5062	0.5063	0.5064	0.5064	0.0792	0.0763	0.0763	0.0762	0.0762
$Z = 0.5$	0.2569	0.2645	0.2648	0.2649	0.2649	0.0626	0.0582	0.0581	0.0580	0.0580
$Z = 0.9$	0.1354	0.1496	0.1500	0.1501	0.1501	0.3438	0.3289	0.3284	0.3282	0.3282

Table 1d
Convergence behavior of moisture content and temperature for $Bi_{qr} = 1$ and $\tau = 0.5$

N	$\theta(Z,\tau)$					$\phi(Z,\tau)$				
	10	100	200	300	330	10	100	200	300	330
$Z = 0.1$	0.8284	0.8343	0.8345	0.8346	0.8346	0.2178	0.2136	0.2134	0.2134	0.2134
$Z = 0.5$	0.5843	0.5896	0.5898	0.5898	0.5898	0.2220	0.2150	0.2147	0.2146	0.2146
$Z = 0.9$	0.4354	0.4424	0.4424	0.4424	0.4424	0.4811	0.4656	0.4650	0.4648	0.4648

Table 1e
Convergence behavior of moisture content and temperature for $Bi_{qr} = 10$ and $\tau = 0.2$

N	$\theta(Z,\tau)$					$\phi(Z,\tau)$				
	10	100	200	250	300	10	100	200	250	300
$Z = 0.1$	0.8503	0.8541	0.8542	0.8542	0.8542	0.0739	0.0720	0.0719	0.0719	0.0719
$Z = 0.5$	0.6244	0.6291	0.6292	0.6292	0.6292	0.0741	0.0710	0.0709	0.0708	0.0708
$Z = 0.9$	0.4215	0.4313	0.4313	0.4313	0.4313	0.3337	0.3230	0.3226	0.3225	0.3225

Table 1f
Convergence behavior of moisture content and temperature for $Bi_{qr} = 10$ and $\tau = 0.5$

N	$\theta(Z,\tau)$					$\phi(Z,\tau)$				
	10	100	200	250	300	10	100	200	250	300
$Z = 0.1$	1.0922	1.0948	1.0948	1.0948	1.0948	0.2221	0.2192	0.2191	0.2191	0.2191
$Z = 0.5$	0.8671	0.8692	0.8692	0.8691	0.8691	0.2438	0.2392	0.2390	0.2389	0.2389
$Z = 0.9$	0.6519	0.6564	0.6563	0.6562	0.6562	0.4736	0.4638	0.4634	0.4632	0.4632

Table 2a
Comparison of one- and two-dimensional solutions of temperature and moisture content for $Bi_{qr} = 0.1$ and $\tau = 0.2$

Solution	$\bar{\theta}(Z,\tau)$			$\bar{\phi}(Z,\tau)$		
	Lumped analysis	$H_{1,1}/H_{0,0}$	2D	Lumped analysis	$H_{1,1}/H_{0,0}$	2D
$Z = 0.1$	0.3029[a]	0.3021[a]	0.3015[b]	0.07721[a]	0.07721[a]	0.07693[b]
	0.3022[b]	0.3014[b]		0.07693[b]	0.07693[b]	
$Z = 0.5$	0.06111	0.06031	0.06051	0.05171	0.05169	0.05148
	0.06124	0.06043		0.05149	0.05147	
$Z = 0.9$	0.0008744	0.0002883	0.0006745	0.3324	0.3324	0.3311
	0.001207	0.0006191		0.3311	0.3311	

[a] Present GITT solution.
[b] NDSolve function (*Mathematica* 5.2) [23].

Table 2b
Comparison of one- and two-dimensional solutions of temperature and moisture content for $Bi_{qr} = 0.1$ and $\tau = 0.5$

Solution	$\bar{\theta}(Z,\tau)$			$\bar{\phi}(Z,\tau)$		
	Lumped analysis	$H_{1,1}/H_{0,0}$	2D	Lumped analysis	$H_{1,1}/H_{0,0}$	2D
$Z = 0.1$	0.5366[a]	0.5352[a]	0.5337[b]	0.1998[a]	0.1997[a]	0.1991[b]
	0.5355[b]	0.5341[b]		0.1990[b]	0.1990[b]	
$Z = 0.5$	0.3118	0.3105	0.3098	0.1963	0 .1963	0.1956
	0.3114	0.3101		0.1956	0.1955	
$Z = 0.9$	0.2616	0.2608	0.2603	0.4760	0 .4761	0.4745
	0.2612	0.2603		0.4743	0.4742	

[a] Present GITT solution.
[b] NDSolve function (*Mathematica* 5.2) [23].

Table 2c
Comparison of one- and two-dimensional solutions of temperature and moisture content for $Bi_{qr} = 1.0$ and $\tau = 0.2$

Solution	$\bar{\theta}(Z,\tau)$			$\bar{\phi}(Z,\tau)$		
	Lumped analysis	$H_{1,1}/H_{0,0}$	2D	Lumped analysis	$H_{1,1}/H_{0,0}$	2D
$Z = 0.1$	0.5546[a]	0.5064[a]	0.5134[b]	0.07591[a]	0.07624[a]	0.07579[b]
	0.5539[b]	0.5058[b]		0.07565[b]	0.07599[b]	
$Z = 0.5$	0.3138	0.2649	0.2734	0.05962	0.05803	0.05825
	0.3139	0.2649		0.05941	0.05778	
$Z = 0.9$	0.1865	0.1501	0.1571	0.3272	0.3282	0.3272
	0.1868	0.1503		0.3259	0.3269	

[a] Present GITT solution.
[b] NDSolve function (*Mathematica* 5.2) [23].

Table 2d
Comparison of one- and two-dimensional solutions of temperature and moisture content for $Bi_{qr} = 1.0$ and $\tau = 0.5$

Solution	$\bar{\theta}(Z,\tau)$			$\bar{\phi}(Z,\tau)$		
	Lumped analysis	$H_{1,1}/H_{0,0}$	2D	Lumped analysis	$H_{1,1}/H_{0,0}$	2D
$Z = 0.1$	0.8885[a]	0.8346[a]	0.8339[b]	0.2156[a]	0.2134[a]	0.2126[b]
	0.8875[b]	0.8336[b]		0.2150[b]	0.2127[b]	
$Z = 0.5$	0.6423	0.5898	0.5909	0.2184	0.2146	0.2144
	0.6418	0.5894		0.2177	0.2139	
$Z = 0.9$	0.4783	0.4424	0.4440	0.4631	0.4648	0.4639
	0.4778	0.4420		0.4613	0.4632	

[a] Present GITT solution.
[b] NDSolve function (*Mathematica* 5.2) [23].

Table 2e
Comparison of one- and two-dimensional solutions of temperature and moisture content for $Bi_{qr} = 10.0$ and $\tau = 0.2$

Solution	$\bar{\theta}(Z,\tau)$			$\bar{\phi}(Z,\tau)$		
	Lumped analysis	$H_{1,1}/H_{0,0}$	2D	Lumped analysis	$H_{1,1}/H_{0,0}$	2D
$Z = 0.1$	1.082[a] 1.082[b]	0.8542[a] 0.8537[b]	0.8598[b]	0.05509[a] 0.05499[b]	0.07195[a] 0.07172[b]	0.06744[b]
$Z = 0.5$	0.9368 0.9368	0.6292 0.6292	0.6526	0.08548 0.08528	0.07084 0.07062	0.07276
$Z = 0.9$	0.7359 0.7358	0.4313 0.4314	0.4648	0.3297 0.3286	0.3225 0.3213	0.3252

[a] Present GITT solution.
[b] NDSolve function (*Mathematica* 5.2) [23].

Table 2f
Comparison of one- and two-dimensional solutions of temperature and moisture content for $Bi_{qr} = 10.0$ and $\tau = 0.5$

Solution	$\bar{\theta}(Z,\tau)$			$\bar{\phi}(Z,\tau)$		
	Lumped analysis	$H_{1,1}/H_{0,0}$	2D	Lumped analysis	$H_{1,1}/H_{0,0}$	2D
$Z = 0.1$	1.097[a] 1.097[b]	1.095[a] 1.094[b]	1.064[b]	0.1856[a] 0.1852[b]	0.2191[a] 0.2185[b]	0.2106[b]
$Z = 0.5$	0.9684 0.9684	0.8691 0.8689	0.8575	0.2511 0.2504	0.2389 0.2383	0.2386
$Z = 0.9$	0.8166 0.8161	0.6562 0.6559	0.6647	0.4918 0.4895	0.4632 0.4617	0.4690

[a] Present GITT solution.
[b] NDSolve function (*Mathematica* 5.2) [23].

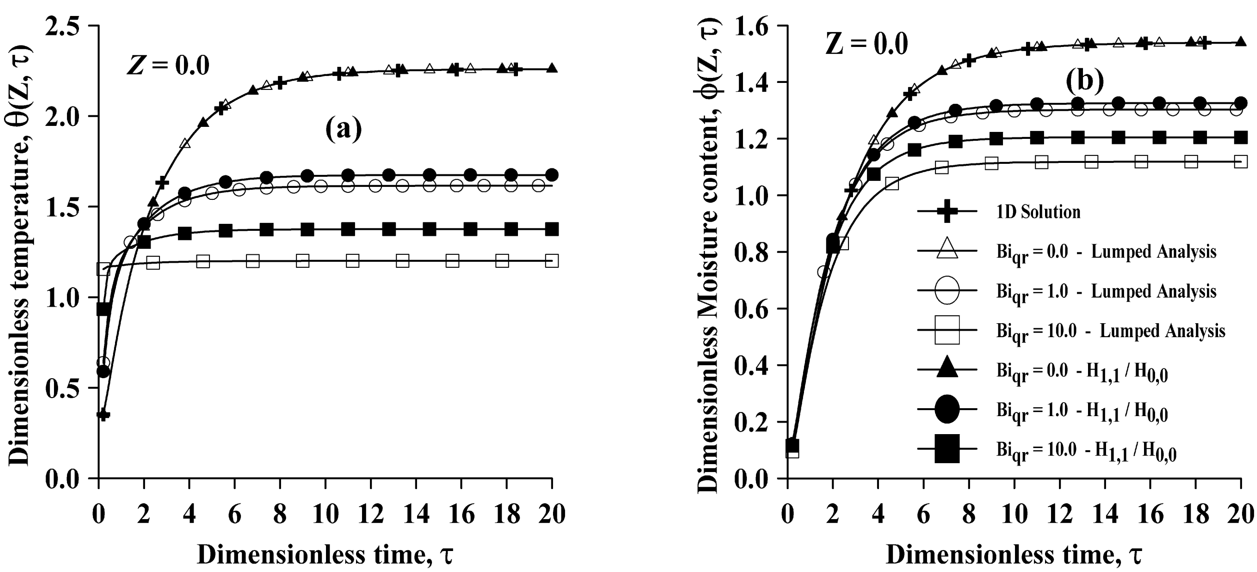

Fig. 1. Comparison of lumped, improved lumped and one-dimensional solutions for (a) temperature and (b) moisture potential profiles with different thermal Biot numbers ($Z = 0$, $Lu = 0.4$, $Pn = 0.6$, $Ko = 5.0$, $Bi_q = Bi_m = 2.5$, $\varepsilon = 0.2$ and $Q = 0.9$).

that bring first a covalidation of the present integral transform analysis with the numerical scheme implemented in the function **NDSolve** of the *Mathematica* system [23], in its default usage mode, following the method of lines. We thus show the GITT solutions for both the classical and improved lumped models, the numerical **NDSolve** solutions for both models, as well as the two-dimensional numerical solution obtained via **NDSolve**. It should be recalled that the present two-dimensional solution is not an actual benchmark result. However, it serves the purpose of inspecting and comparing the two approximate formulations, the classical lumped formulation and the improved lumped formulation here proposed, since the actual two-dimensional formulation in Eqs. (5) is being dealt with by the **NDSolve** routine. It can be observed that the default numerical solution implemented in [23], reproduces to plus or minus one in the third digit given in Tables 2, the present error

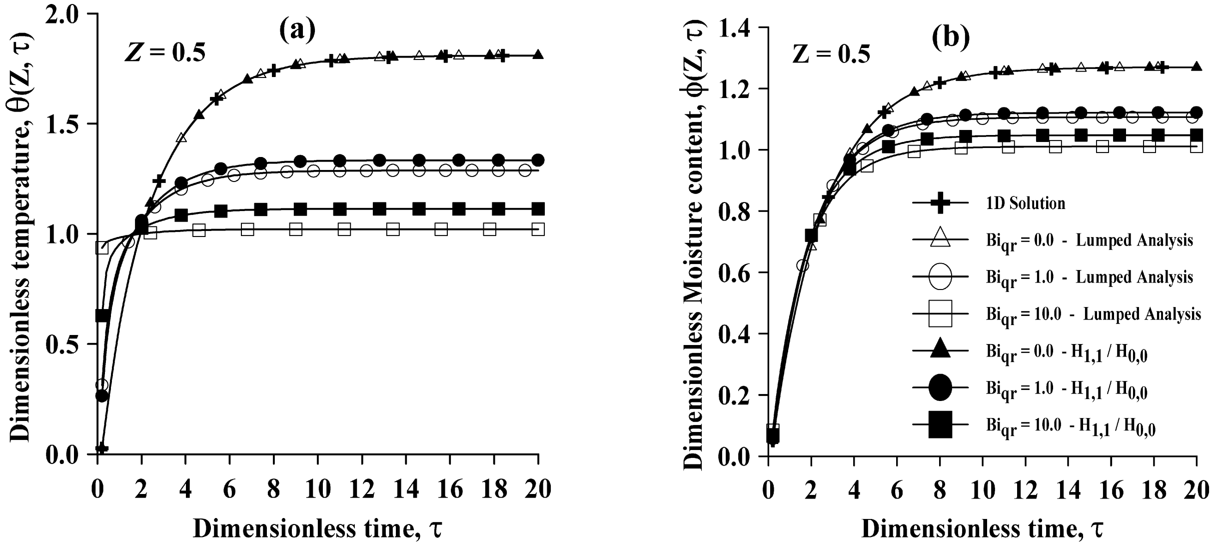

Fig. 2. Comparison of lumped, improved lumped and one-dimensional solutions for (a) temperature and (b) moisture potential profiles with different thermal Biot numbers ($Z = 0.5$, $Lu = 0.4$, $Pn = 0.6$, $Ko = 5.0$, $Bi_q = Bi_m = 2.5$, $\varepsilon = 0.2$ and $Q = 0.9$).

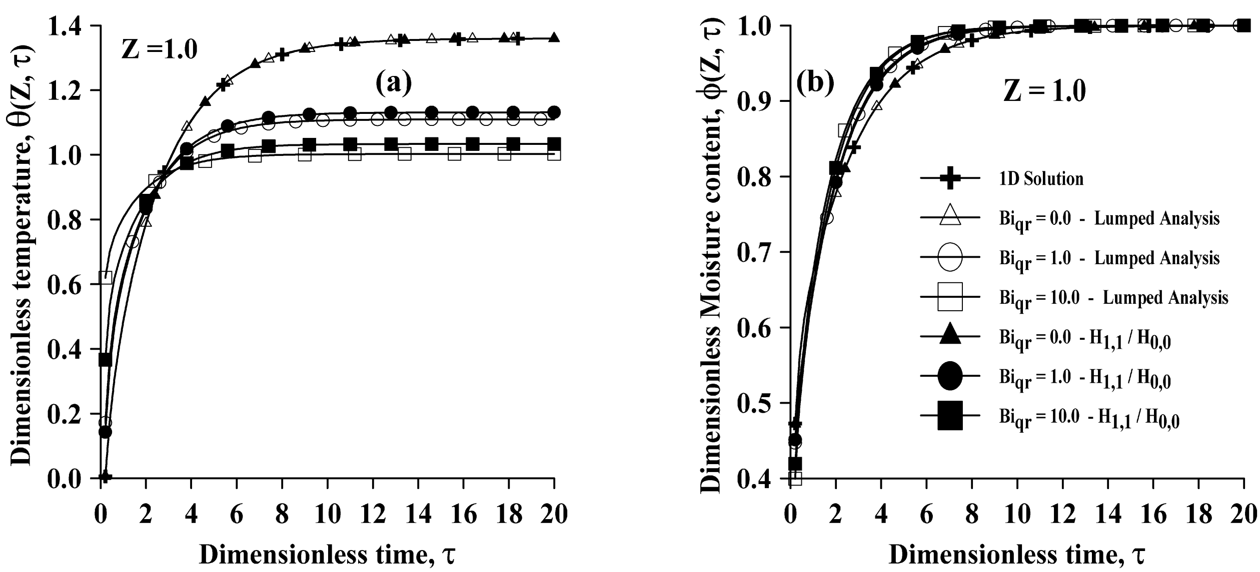

Fig. 3. Comparison of lumped, improved lumped and one-dimensional solutions for (a) temperature and (b) moisture potential profiles with different thermal Biot numbers ($Z = 1.0$, $Lu = 0.4$, $Pn = 0.6$, $Ko = 5.0$, $Bi_q = Bi_m = 2.5$, $\varepsilon = 0.2$ and $Q = 0.9$).

controlled solution, for either the classical and improved one-dimensional lumped models. Second, the direct comparison of the one-dimensional and two-dimensional solutions allow for the evaluation of the approximation improvement achieved by the proposed model, in comparison with the classical lumping approach. The results for increasing radial Biot number values, and more impressively for the largest value in Tables 2e, 2f ($Bi_{qr} = 10$), illustrate the significant loss of accuracy experienced by the classical lumped solution, while the improved one-dimensional model still retains the same behavior of the full two-dimensional model, with acceptable accuracy for most applications requirements. The comparison of the two approximate solutions proposed here with such a 2D solution identifies ranges of validity in terms of the radial Biot number, for different values of $Lu, Pn, Ko, Bi_q, Bi_m, Q$ and ε.

Figs. 1–3 graphically illustrate the comparison of the one-dimensional models, classical and improved, as well as the results obtained with the exact one-dimensional solution *via* GITT [15] ($Bi_{qr} = 0$), for the average temperature and average moisture content at the positions $Z = 0$, 0.5 and 1, respectively, with $Lu = 0.4$, $Pn = 0.6$, $Ko = 5.0$, $Bi_q = Bi_m = 2.5$, $\varepsilon = 0.2$ and $Q = 0.9$. As expected, the lumped and CIEA solutions are in perfect agreement with the 1D solution for $Bi_{qr} = 0$. As Bi_{qr} increases, the average temperatures obtained with the approximate solutions tend to be smaller than that for the 1D solution, due to the lateral heat losses. The same behavior is observed for the average moisture content. By comparing the two approximate solutions, we can notice that the average temperatures and the average moisture contents tend to be larger with the $H_{1,1}/H_{0,0}$ approximation than with the lumped approach, for

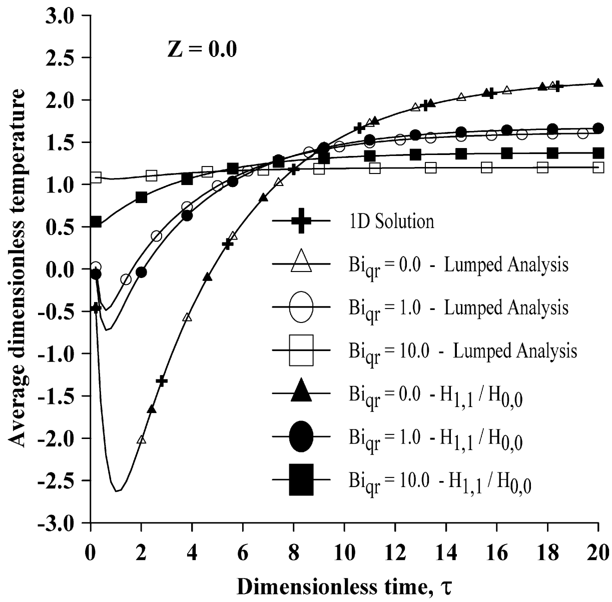

Fig. 4. Comparison of lumped, improved lumped and one-dimensional solutions for temperature profiles with different thermal Biot numbers ($Z = 0$, $Lu = 0.2$, $Pn = 0.084$, $Ko = 49$, $Bi_q = Bi_m = 2.5$, $\varepsilon = 0.2$ and $Q = 0.9$).

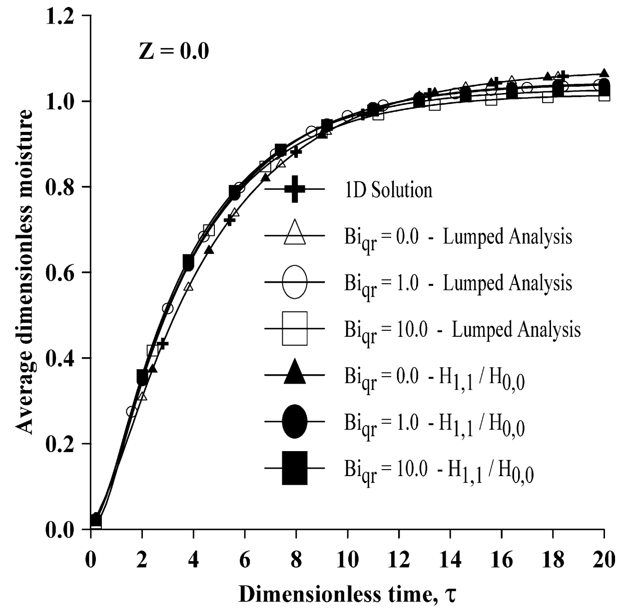

Fig. 5. Comparison of lumped, improved lumped and one-dimensional solutions for moisture potential profiles with different thermal Biot numbers ($Z = 0$, $Lu = 0.2$, $Pn = 0.084$, $Ko = 49$, $Bi_q = Bi_m = 2.5$, $\varepsilon = 0.2$ and $Q = 0.9$).

most of the time domain analyzed. This is due to the fact that the modified Biot number given by Eq. (9) for the $H_{1,1}/H_{0,0}$ approximation, which takes into account the radial gradients, is smaller than the actual Biot number. This effect is more noticeable for larger Biot numbers in the radial direction, such as $Bi_{\mathrm{qr}} = 10$. We may also observe from Figs. 2 and 3 for the positions $Z = 0.5$ and 1, respectively, further away from the heated boundary, that the variations on average temperature and moisture, due to the lateral heat losses, are less marked than those at the position $Z = 0$ as shown in Fig. 1. An interesting behavior is shown in Fig. 3(b), as the drying process evolves to large dimensionless times, the moisture content at the upper surface ($Z = 1$) reaches the equilibrium moisture value of the external air, and this result is not significantly affected by the lateral heat losses.

After analyzing Figs. 1–3 for the parameters $Lu = 0.4$, $Pn = 0.6$, $Ko = 5.0$, $Bi_q = Bi_m = 2.5$, $\varepsilon = 0.2$ and $Q = 0.9$, it is now of interest to analyze the case for which negative dimensionless temperatures are observed due to vaporization within the medium. We thus take the case of $Lu = 0.2$, $Pn = 0.084$, $Ko = 49$, $Bi_q = Bi_m = 2.5$, $\varepsilon = 0.2$ and $Q = 0.9$, and the related results for the average temperature and moisture content for $Z = 0$ are presented in Figs. 4 and 5, respectively. One can notice from Fig. 4 that the lateral heat losses tend to diminish the vaporization effect, and consequently, to increase the average temperature. The temperature raise in this case, in which the water vaporization is important, follows the opposite trend as observed in Fig. 1(a), for the case when negative dimensionless temperatures are not observed, for the small dimensionless times, due to vaporization. It can be observed from Fig. 5 that the moisture potential profiles at $Z = 0$ are not markedly affected by the lateral heat losses, as opposed to the significant variation for the case in Fig. 1(b).

5. Conclusions

The use of hybrid tools in formulation, solution and computation of simultaneous heat and mass transfer problems has been discussed and illustrated. The Coupled Integral Equations Approach has been recently employed to provide *a priori* error analysis [9], with encouraging results, and should be progressively extended to more complex nonlinear formulations. The Generalized Integral Transform Technique enters now a phase of algorithm refinement and optimization, which includes advanced filtering and reordering schemes, enhanced approaches for ODE systems, and automatic implementation for arbitrarily irregular geometries. The *Mathematica* system has been intensively employed in conjunction with the approaches above described, aimed at further facilitating the analytical development task [22].

The numerical modeling of drying in capillary porous media requires the accurate knowledge of several thermophysical and boundary condition parameters that appear in the formulation. The use of inverse analysis techniques permits the estimation of several of such parameters, from the knowledge of temperature and moisture content measurements taken in the media. The direct problem solution paths above discussed may also be employed in different inverse problem analysis for drying of capillary porous media [25–29]. The present approach also adds up to a class of simplification tools related to model reduction [30,31], and their combined use might offer novel solution paths with further reduction in computational cost.

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